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Experimental Tests of a Laminar Mixing Theory

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A recently developed analysis of the laminar mixing of two steady two-dimensional flows is subjected to a comparison with two different sets of experimental data. The theory considers the mixing beyond the trailing edge of a thin partition, accounts for the initial boundary layers, and allows broad dissimilarities in the two Mach numbers and total temperatures. Earlier comparisons produced good agreement between this theory and numerical and experimental results for wakes and base flows. In this paper, the theory is compared with the measured flowfield of a mixing layer separating two isothermal streams at nominal Mach numbers 3 and 2.3. A second experiment is also reported here, in which a Mach 8 flow merges with a colder Mach 3 stream. In both cases the agreement between theory and experiment is satisfactory.

Nomenclature

a = speed of sound
a' = see Eq. (8)
b' = see Eq. (8)
C_n = numerical constants in Eq. (5), see Ref. 1
M = Mach number
M_c = convective Mach number, $(u_1 - u_2)/(a_1 + a_2)$
M_e = Mach number at the edge of viscous flow
P' = asymmetry parameter, Eq. (4)
Re' = unit Reynolds number, $\rho_1 u_1/\mu_1$
$Re_{\theta 1}$ = momentum Reynolds number, $u_1\rho_1\theta_1/\mu_1$
Re_{XT} = transition Reynolds number, $u_1\rho_1 XT/\mu_1$
S = profile stretching factor
T = static temperature
T_0 = total (stagnation) temperature
T_W = partition temperature
u = longitudinal velocity
$\bar{u}_0$ = profile $1 - (u/u_1)$ at $x^* = 0$
XT = distance from $x^* = 0$ to transition
x^* = longitudinal coordinate, Fig. 1
x' = dimensionless form of x^*, Eq. (6)
$\bar{x}$ = $x^*/\theta_1 Re_{\theta 1}$
y = lateral coordinate, Eq. (3)
y^* = lateral coordinate, Fig. 1
$\bar{y}^*$ = compressible-transformed form of y^*, Eq. (3)
y'' = see Eq. (7)
y'_m = value of y'' at wake minimum
γ = specific heat ratio
η = see Eq. (9)
θ = boundary-layer momentum thickness at $x^* = 0$
μ = viscosity
ρ = density

Subscripts
1 = constant pertaining to faster stream
2 = constant pertaining to slower stream

I. Introduction

A RECENT theory[1] calculates analytically the flowfield of two laminar, steady coflowing streams merging downstream of the trailing edge of a thin partition. The analysis provides closed-form solutions over a very wide range of physical coordinates and flow conditions, and should therefore be of practical use in film-cooling design and other applications, including an understanding of the processes inside hypersonic airbreather combustors. For supersonic and hypersonic flows, especially, laminar mixing is relevant since turbulent transition is delayed because of increased stability of the upstream boundary layers,[2] and of the shear layers downstream of the merging point.[3]

In Ref. 1 the theory is compared favorably with numerical computations, and with measurements, in wakes and base flows. In this paper, the comparison is extended to cases involving two dissimilar coflowing streams. In general, such flows are asymmetric in the inviscid sense [conditions $()_1 \neq$ conditions $()_2$] and also in the viscous sense (e.g., $\theta_1 \neq \theta_2$). Wake-like behavior appears only near the trailing edge, while at asymptotically large x^*, in the nomenclature of Fig. 1, the flowfield relaxes to the classic free-shear-layer profile.[4] This relaxation distance is of some practical importance because stability calculations, as an example, often simplify the laminar mixing profile by some convenient approximation, e.g., the hyperbolic tangent.[5,6] The present theory could, if valid, show whether such an approximation is justified, provide preasymptotic profiles in cases where the relaxation distance is long, or rule in favor of calculations based on numerical solutions of the mean motion.[7]

II. Review of the Theory

The analysis[1] in question addresses the steady, laminar, two-dimensional mixing of two chemically homogeneous, nonreacting ideal fluid streams, as per Fig. 1. A unity Prandtl number and a constant Chapman-Rubesin factor are assumed

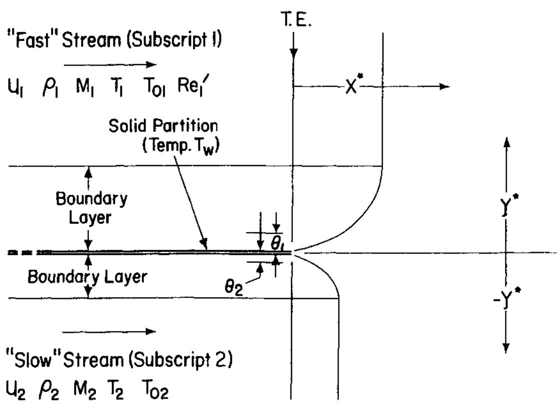

Fig. 1 Flowfield geometry and nomenclature for the theory of Ref. 1.

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throughout. The momentum thicknesses of the two boundary layers at the trailing edge of the partition are θ_1 and θ_2, respectively, and their velocity profiles are assumed to have the following form:

$$\text{Stream 1} \quad (y^*>0): \bar{u}_0 = e^{-(y/S)} \tag{1}$$

$$\text{Stream 2} \quad (y^*<0): \bar{u}_0 = e^{(P'y/S)} \tag{2}$$

where

$$y \equiv \bar{y}^*/\theta_1 = \frac{1}{\theta_1}\int_0^{y^*}\frac{\rho}{\rho_1}\,\mathrm{d}y^* \tag{3}$$

$$P' \equiv \theta_1\rho_1/\theta_2\rho_2 \tag{4}$$

and where S is a numerical "stretching" factor which can be chosen by the user to simulate the trailing-edge boundary-layer velocity profiles (e.g., $S = 3$ approximates the Blasius profile). For this flow, Ref. 1 provides closed-form solutions for the flow velocity and temperature in the region $0<x^*<\infty$ once the parameters Re_1', M_1, T_{02}/T_{01}, T_W/T_{01}, γ, P', and S are chosen. In the nomenclature of Fig. 1, the solution for the local velocity is

$$\frac{u}{u_1} = \left\{1 - \frac{1}{2}\frac{e^{-y''^2}}{4x'}\left[e^{a'^2}\operatorname{erfc}(a') + e^{b'^2}\operatorname{erfc}(b')\right]\right\} \times\left[\left(1-\frac{u_2}{u_1}\right)\sum_0^6 C_n\eta^n + \frac{u_2}{u_1}\right] \tag{5}$$

$$x' \equiv x^*/(\theta_1+\theta_2)^2 Re_1' \tag{6}$$

$$y'' \equiv 1/S[\bar{y}^*/(\theta_1+\theta_2) + (S-1)y_m'] \tag{7}$$

$$a' \equiv (P'+1)\sqrt{x'} + y''/(2\sqrt{x'})$$

$$b' \equiv [(P'+1)/P']\sqrt{x'} - y''/(2\sqrt{x'}) \tag{8}$$

$$\eta \equiv 1/\sqrt{x'}[\bar{y}^*/(\theta_1+\theta_2) - y_m'] \tag{9}$$

Reference 1 describes how this formula, and one analogous for the local temperature, is obtained by considering the laminar mixing layer as a superposition ("composite") of two "components," a wake component caused by the shed boundary layers and analyzed by the classic boundary-layer equations, and a classic free-shear-layer component as treated by Chapman,[4] caused by the inviscid asymmetry $(\,)_1 \neq (\,)_2$. The quantity y_m' featured in the preceding equations is essentially the value of y'' for which the wake-component velocity [the first bracket in Eq. (5)] is a minimum. The numerical constants C_n, presented in Ref. 1, provide a very accurate reproduction of Chapman's quadrature solution for the velocity in the free shear layer, which is the second bracketed term in Eq. (5).

Table 1 Experimental parameters

	Supersonic mixing (Brower)	Hypersonic mixing (Present work)
M_1	3	8
M_2	2.3	3
u_2/u_1	0.89	0.585
γ	1.4	1.4
θ_1, cm	0.0123	0.0207
θ_2, cm	0.0152	0.0195
T_{01}, K	336	723
T_{02}, K	336	359
T_W, K	336	524
P'	1.11	2.61
S	1.81	2
M_c	0.228	1.31

This solution is not interrupted by singularities or discontinuities downstream of the trailing edge and can simulate a wide variety of flows by the appropriate choice of the input parameters listed previously. Parametric effects can be evaluated quickly without carrying out numerical solutions for each case separately, and good comparisons have already been demonstrated[8] with such existing numerical solutions (and experimental data) for wakes $[(\,)_1 = (\,)_2]$ and base flows $(u_2 = 0)$. In this paper, we seek to extend the comparison with test data on coflowing streams. Such data must deal only with shear layers which are documentedly laminar, and must contain quantitative information on all of the parameters mentioned as necessary theoretical inputs. An experiment at supersonic speeds meeting these requirements is the work of Brower[9,10]; to cover higher Mach numbers, a new experiment was performed in the present work, which will be described in Sec. IV.

III. Supersonic Mixing Data

Brower[9,10] has provided detailed flow maps for two laminar streams supersonic relative to, and mixing downstream of, a thin partition. The relevant parameters of this experiment, performed in the Montana State University continuous supersonic wind tunnel and conforming to the geometry of Fig. 1, are listed in Table 1. This table also lists the so-called convective Mach number M_c for this experiment. In addition, the measurements providing the listed parameters included a measurement of the transition distance from the trailing edge, made with Schlieren photography and hot-wire anemometry. Brower found transition in the free shear layer located a few centimeters downstream of his partition trailing edge. His

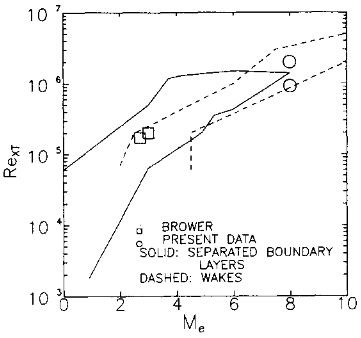

Fig. 2 Transition Reynolds numbers for the two experiments under comparison with the theory.

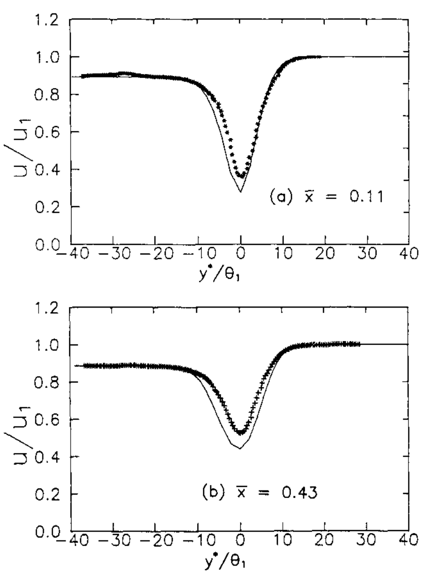

Fig. 3 Comparison of the velocity data (points) with the theory of Eq. (5) (solid curve) in the case of supersonic mixing.

computed transition Reynolds number Re_{XT}, shown in Fig. 2, agrees with other observations of Re_{XT} in free shear flows such as wakes and separated boundary layers.

Several lateral flow profiles (along y^*) were obtained by Brower in the laminar flow between his trailing edge and the transition region. To compare these data with the theory of Sec. II, Eq. (5) was evaluated at each x^* at which data had been obtained, using as inputs the parameters of Table 1. For convenience, the nondimensional space coordinates corresponding to x^* and y^* were computed in the form of $\bar{x}$ and y^*/θ_1, instead of the x' and y'' given in Eqs. (6) and (7). The stretching factor S which, as indicated earlier, allows theoretical simulation of the experimental trailing-edge (initial) velocity profile, was found by fitting the data points (u,y^*) into an exponential according to the assumptions of the theory [i.e., Eq. (1)] and finding the S for which the fit was best. As Fig. 4 of Ref. 1 indicates, Brower's laminar boundary-layer profiles were fitted very well by the exponential of Eqs. (1) and (2) if $S = 1.81$. Such a value of S is indicative of an accelerating flow and is ascribed to the favorable pressure gradient in the deLaval nozzle, as was confirmed by Brower by surface pressure measurements.[9]

Figures 3 and 4 show the velocity profiles in the laminar mixing layer with Brower's data represented by points compared with the theory of Eq. (5). The $\bar{x}$ range shown is necessarily short in order to avoid including transitional and turbulent data, and therefore both the theory and the data show a preponderance of the wake component of the flow (the "dip" of the velocity in the center). In this range, which covers a variation of $\bar{x}$ from 0 to about 0.43, the theory agrees qualitatively with the experiment but seems to consistently underestimate the flow velocity somewhat; the minimum theoretical profile velocity, according to Fig. 4, is of order 18% lower than that found in the measurement.

The noted discrepancies are not readily explainable, although there are possibilities regarding the theory and the experiment as well. A Prandtl number different from unity would perhaps be an obvious improvement of the laminar mixing analysis presented here. On the side of the experiment, the method of normalizing the coordinates by θ_1 places a serious burden on the experimental accuracy requirement, since an error in measuring θ_1 would have a proportional effect on the points plotted in Figs. 3 and 4. For example if θ_1 was actually about 25% smaller than reported by Brower, then the data points on Fig. 4 would be shifted to the right sufficiently to agree exactly with the theory. Such an error in the measurement of θ_1 might have been present in Brower's measurements, which involved the very delicate task of probing a 0.2-cm-thick boundary layer with a 0.008-cm-diam pitot probe. Efforts made in the present work to make such after-the-fact "corrections" to Brower's data were inconclusive, however, and the status of the comparison with theory therefore remains as on Figs. 3 and 4.

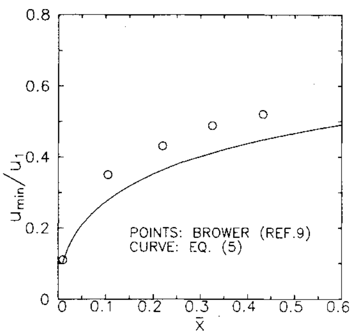

Fig. 4 Comparison of the experimental minimum velocities at each x^* (points) with the predictions of Eq. (5) (solid curve) for the supersonic mixing case.

IV. Hypersonic Mixing Experiment

The second experiment listed on Table 1, performed during the present work, concerns a laminar shear layer generated by the merging of a Mach 8 stream (subscript 1) with another at Mach 3 (subscript 2). These measurements were conducted in Hypersonic Wind-Tunnel B of the Arnold Engineering Development Center (AEDC). In this experiment, a sharp-lipped flat plate was inserted at zero incidence in the steady-state Mach 8 flow of this tunnel. The data shown here were taken at a tunnel stagnation pressure of 200 psia (1.38×10^6 N/m^2) and temperature of 723 K, giving a stream unit Reynolds number of 32,800/cm. The plate had a width of 30.5 cm and an overall length of 86.4 cm. A 1.35-cm-tall backward-facing step was located on the plate surface, 48 cm downstream of the leading edge. A steady two-dimensional air jet at exit Mach number 3 emerged from a nozzle at the base of the step, as shown on Fig. 5. The Mach 3 nozzle was designed by the characteristics method, and its exit height equaled the step height; the solid structure separating the Mach 3 and Mach 8 flows thus formed a thin partition with a sharp trailing edge. An internal manifold provided air to the jet at a stagnation temperature of 359 K, and at a stagnation pressure set to keep the jet exit pressure equal to the static pressure of the tunnel stream.

At the flow conditions stated, the initial boundary layers were laminar on both sides of the partition trailing edge (at $x^* = 0$). Schlieren photography and hot-wire and hot-film anemometry were used to find the location XT, downstream of the trailing edge, where the free shear layer became turbulent. Transition occurred at about $x^* = 28$ cm ($XT = 28$ cm, $\bar{x} = 2.2$), yielding a Re_{XT} of just under 1×10^6. This datum is shown on Fig. 2, accompanied by another at about 2×10^6, also obtained in these tests at a unit Reynolds number of 131,200/cm. According to Fig. 2, these results are again consistent with data from other free shear flows. Note, too, that in this experiment the transition distance XT was much larger than in Brower's case, corresponding to a similar difference, according to Table 1, in the convection Mach number M_c.

In contrast to the supersonic mixing experiment, in this case the slower Mach 3 flow was limited in the lateral $-y^*$ direction by the flat wall following the step (Fig. 5). By carefully balancing the pressures of the two streams, it was possible to maintain the mixing layer parallel to this wall with only a small displacement, caused by the wall boundary-layer growth, of the former toward $+y^*$. As the figures to be discussed shortly will show, however, the inviscid core of the Mach 3 stream maintained its identity for a considerable distance, and the displacement was fully accounted for in comparing the data with the theory.

Seven lateral flow profiles were obtained in the laminar portion of the shear layer, at various distances x^*, with the use of pitot probes, calibrated recovery-temperature thermocouples, and static pressure orifices located on the plate following the step; the latter gave slightly anomalous readings for a length equaling one step height after the jet exit, but reached the Mach 3 (and Mach 8) static pressure soon afterwards. The partition temperature T_W was measured with a thermocouple. Additional thermocouples and heat transfer gauges located on

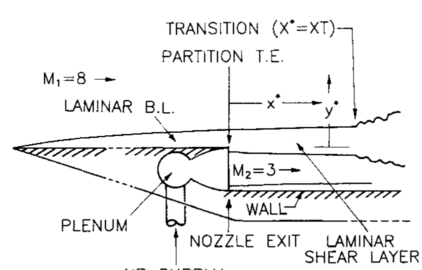

Fig. 5 Schematic of the test setup for the hypersonic mixing experiment.

the surface following the step were used to verify that the shear layer flow was two-dimensional.

V. Hypersonic Mixing Data

The data just described was used to find the profiles of velocity, temperature, etc. and to provide the parameters, listed on Table 1, for calculating the velocity theoretically from Eq. (5). The appropriate stretching parameter S for the initial fast-side boundary layer at the trailing edge was again found by first curve fitting the velocity-profile data including the $u(x^* = y^* = 0) = 0$ point, and then searching for the S which would bring this curve fit closest to Eqs. (1) and (2). An excellent fit was found for $S = 2$, from which $\theta_1 = 0.00815$ in. ($= 0.0207$ cm) was obtained. Because of resolution problems with the extremely thin boundary layer, the slow-side momentum thickness θ_2 could not be measured but was computed theoretically, using the test data only as a cross-check.

Figures 6–8 show the results of the comparison of these data with the theory. The boundary layer seen on the left of Figs. 6 and 7, which in the test grew on the solid wall following the step, must be noted; this boundary layer is of course not included in the analysis and is thus irrelevant to the present issue. Qualitatively the agreement for the mixing layer is very good; even at a distance of $x^* = 22.9$ cm the wake component of the flow, represented by the "dip" of the velocity at the center, is distinct both in the theory and in the experiment. As before, and probably for the same reasons, the theory underestimates the velocity (and overestimates the temperature according to Fig. 7) somewhat, especially on the fast-stream side, giving a "thicker" shear layer than the data indicate. On the other hand, the minimum-velocity prediction, according to Fig. 8, is much better than found for the supersonic mixing case. Further details of these comparisons are contained in Ref. 11.

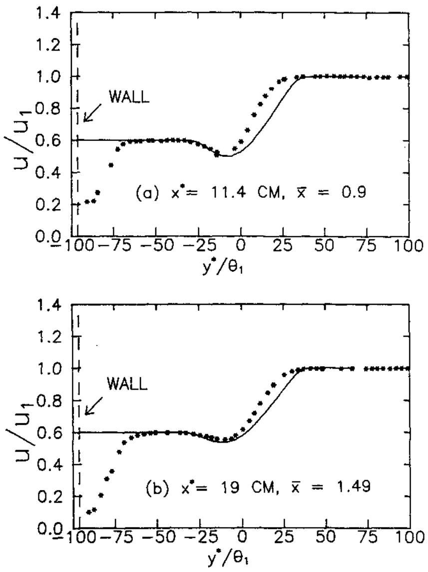

Fig. 6 Comparison of the velocity data (points) with the theory of Eq. (5) (solid curve) in the case of hypersonic mixing.

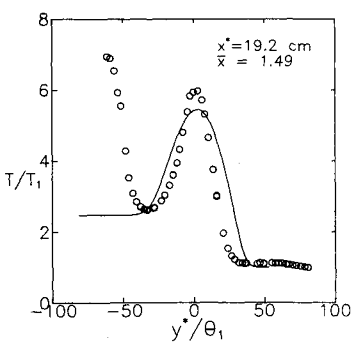

Fig. 7 Comparison of the measured temperatures at a typical downstream station (points) with the present theory (solid curve) in the case of hypersonic mixing.

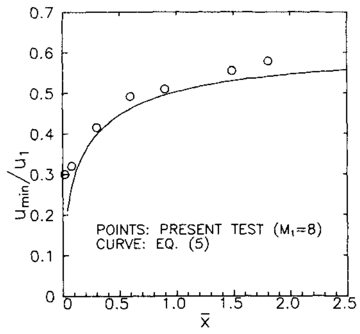

Fig. 8 Comparison of the experimental minimum velocities (points) with the predictions of Eq. (5) (solid curve) for hypersonic mixing.

VI. Concluding Remarks

The "composite" or "superposition" method of predicting laminar mixing according to Ref. 1 has been tested here for the first time against experimental data on two coflowing ($u_1 \neq u_2 \neq 0$) high-speed streams. The principal characteristics of the theoretical predictions, e.g., the persistence of the wake component of the mixing layer and the rate of decay of the minimum velocity, were in qualitative, and often in good quantitative, agreement with the data.

The most obvious shortcoming of the theory is a modest overestimate of the mixing-layer thickness and its rate of spreading, attributable to the assumption of unity Prandtl number. It should also be recalled that, aside from this and similar assumptions underlying the theory already expressed in Sec. II, the analysis assumes a strictly two-dimensional flow, infinite width of the two streams in the lateral (y^*) direction, an infinitely thin partition, and initial boundary-layer profiles which can be fitted accurately by an exponential. It is clearly impossible to devise an experiment that will meet these requirements exactly. At the same time, full numerical solutions of the flow geometries discussed here may also be equally impractical in many cases for which the theory of Ref. 1 can serve as an acceptable alternative.

Finally, it must be remarked that in both experiments discussed here, transition to turbulence occurred very shortly beyond the zone covered by the profile data presented in Figs. 3, 6, and 7. Hydrodynamic stability calculations are sometimes done based on simple, asymptotic mixing-layer velocity profiles in order to anticipate transition onset. The theory shows that the relevant laminar profiles are not quite so simple, and could perhaps offer analytical forms for them which could improve the stability analyses.

Acknowledgments

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